A Frobenius manifold for ℓ-Kronecker quiver

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May 24, 2022

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Generalized root system

Definition 1 (Root system).

A (simply-laced) root system of rank μ is a tuple $R = (L, I, \Delta^{\mathrm{re}})$,

- *a free $\mathbb{Z}$-module L of rank $\mu \in \mathbb{Z}_{\geq 0}$,*
- *a symmetric $\mathbb{Z}$-bilinear form $I \colon L \times L \longrightarrow \mathbb{Z}$, called the Cartan form,*
- *a subset $\Delta^{\mathrm{re}} \subset L$ called the set of real roots,*

such that

1. $L = \mathbb{Z}\Delta^{\mathrm{re}}$,
2. $I(\alpha, \alpha) = 2$ *for any* $\alpha \in \Delta^{\mathrm{re}}$,
3. $r_\alpha(\Delta^{\mathrm{re}}) = \Delta^{\mathrm{re}}$ *for any* $\alpha \in \Delta^{\mathrm{re}}$, *where* $r_\alpha \in \mathrm{Aut}_{\mathbb{Z}}(L, I)$ *is a reflection defined by*

$$r_\alpha(\lambda) = \lambda - I(\alpha, \lambda)\alpha, \quad \lambda \in L.$$

The Weyl group is defined to be

$$W := \langle r_\alpha \mid \alpha \in \Delta^{\mathrm{re}} \rangle.$$

Let $R = (L, I, \Delta^{\mathrm{re}})$ be a root system.

Define two $\mathbb{C}$-vector spaces $\mathfrak{h}$ and $\mathfrak{h}^*$ by

$$\mathfrak{h} := \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{C}), \quad \mathfrak{h}^* := L \otimes_{\mathbb{Z}} \mathbb{C},$$

and denote the natural coupling by $\langle -, - \rangle \colon \mathfrak{h}^* \times \mathfrak{h} \longrightarrow \mathbb{C}$.
The $\mathbb{C}$-vector space $\mathfrak{h}$ is called the Cartan subalgebra.

The Weyl group W acts on $\mathfrak{h}$ as follows:

$$\langle \alpha, w(x) \rangle := \langle w^{-1}(\alpha), x \rangle, \quad \alpha \in \mathfrak{h}^*, \ x \in \mathfrak{h}, \ w \in W.$$

We should consider an additional data:

Definition 2 (Generalized root system).

Let $R = (L, I, \Delta^{\mathrm{re}})$ be a root system of rank μ and $B = \{\alpha_1, \ldots, \alpha_\mu\}$ is a root basis. Define a Coxeter element $\mathbf{c} \in W$ by

$$\mathbf{c} := r_{\alpha_1} \cdots r_{\alpha_\mu}.$$

A generalized root system is a pair $(R, \mathbf{c})$ consisting of a root system R and a Coxeter element $\mathbf{c}$.

Remark: Coxeter elements depends on the choice and order of root bases.

The reasons why we consider Coxeter elements are

- Serre functor in the (triangulated) categorical theory,
- Milnor monodromy in the deformation theory and primitive form,
- to obtain Frobenius structures.

Triangulated category

Let $\mathcal{D}$ be a $\mathbb{C}$-linear triangulated category of finite type and $\mathcal{S}$ the Serre functor:

$$\mathrm{Hom}_{\mathcal{D}}(X,Y) \cong \mathrm{Hom}_{\mathcal{D}}(Y,\mathcal{S}(X))^*, \quad X,Y \in \mathcal{D}.$$

We can obtain a generalized root system $(R_{\mathcal{D}}, \mathbf{c}_{\mathcal{D}}) = (L_{\mathcal{D}}, I_{\mathcal{D}}, \Delta^{\mathrm{re}}_{\mathcal{D}}, \mathbf{c}_{\mathcal{D}})$ from a “good” triangulated category $\mathcal{D}$:

Let $(E_1, \ldots, E_\mu)$ be a full strongly exceptional collection and $B = \{[E_1], \ldots, [E_\mu]\}$ a (root) basis on the Grothendieck group $K_0(\mathcal{D})$.

- $L_{\mathcal{D}} := K_0(\mathcal{D})$,
- $I_{\mathcal{D}} := \chi + \chi^T$, where χ is the Euler form,
- $\Delta^{\mathrm{re}}_{\mathcal{D}} := W(B)B$, where $W(B) = \langle r_{[E_i]} \mid i = 1, \ldots, \mu \rangle$,
- $\mathbf{c}_{\mathcal{D}} := -[\mathcal{S}] = -\chi^{-1}\chi^T = r_{[E_1]} \cdots r_{[E_\mu]}$.

The generalized root system $(R, \mathbf{c})$ is independent of the choice of full strongly exceptional collections.

Let Q be a Dynkin quiver.
The derived category $\mathcal{D}^b(Q) := \mathcal{D}^b(\mathrm{mod}\,\mathbb{C}Q)$ is a "good" triangulated category.

Then the generalized root system $(R_{\mathcal{D}}, \mathbf{c}_{\mathcal{D}})$ associated with $\mathcal{D} = \mathcal{D}^b(Q)$ is of type ADE corresponding to Q.

Geometric background

Let $f\colon \mathbb{C}^3 \longrightarrow \mathbb{C}$ be a polynomial of type ADE.

Define $(R_f, \mathbf{c}_f) = (L_f, I_f, \Delta_f^{\mathrm{re}}, \mathbf{c}_f)$ as follows :

- $L_f := H_2(f^{-1}(1), \mathbb{Z})$ is the homology group of Milnor fiber,
- $I_f := -I_{H_2}$, where I_{H_2} is the intersection form on $H_2(f^{-1}(1), \mathbb{Z})$,
- $\Delta_f^{\mathrm{re}} := \{\alpha \in L_f \mid I_f(\alpha, \alpha) = 2\}$,
- $\mathbf{c}_f := \mathbf{h}_f^{-1}$ is the inverse of the Milnor monodromy $\mathbf{h}_f$.

Then $(R_f, \mathbf{c}_f)$ is a generalized root system.

Theorem 3.

Let $f\colon \mathbb{C}^3 \longrightarrow \mathbb{C}$ be a polynomial of type ADE and Q a Dynkin quiver corresponding to f. There exists an isomorphism

$$(R_f, \mathbf{c}_f) \cong (R_{\mathcal{D}}, \mathbf{c}_{\mathcal{D}}),$$

where $\mathcal{D} = \mathcal{D}^b(Q)$.

Definition of Frobenius manifolds

Let M be an μ-dimensional complex manifold.

Definition 4 (Frobenius manifold).

A tuple $(\eta, \circ, e, E)$ *consisting of*

- $\eta : \mathcal{T}_M \times \mathcal{T}_M \longrightarrow \mathcal{O}_M$*: non-degenerate symmetric* $\mathcal{O}_M$*-bilinear form,*
- $\circ : \mathcal{T}_M \times \mathcal{T}_M \longrightarrow \mathcal{T}_M$*: associative commutative* $\mathcal{O}_M$*-bilinear product,*
- $e \in \Gamma(M, \mathcal{T}_M)$*: the unit of* $\circ$*,*
- $E \in \Gamma(M, \mathcal{T}_M)$*, which is called the Euler vector field.*

is called a Frobenius structure of (conformal) dimension $d \in \mathbb{C}$ *on* M *if it satisfies a certain conditions.*

A Frobenius manifold $M = (M, \eta, \circ, e, E)$ *is a complex manifold equipped with a Frobenius structure* $(\eta, \circ, e, E)$*.*

Axioms of Frobenius structures

For any $\delta, \delta', \delta'' \in \mathcal{T}_M$,

1. The Levi-Civita connection $\nabla\!\!\!/ : \mathcal{T}_M \longrightarrow \mathcal{T}_M \otimes \Omega_M^1$ with respect to η is flat,
2. The product $\circ$ is self-adjoint with respect to η: that is,
$$\eta(\delta \circ \delta', \delta'') = \eta(\delta, \delta' \circ \delta''),$$
3. (potentiality condition) The tensor $C : \mathcal{T}_M \longrightarrow \mathrm{End}_{\mathcal{O}_M} \mathcal{T}_M$ defined by $C_\delta \delta' := \delta \circ \delta'$ is $\nabla\!\!\!/$-flat,
4. The unit element e is $\nabla\!\!\!/$-flat,
5. The product $\circ$ and the metric η are homogeneous of degree 1 and $2-d$, respectively, with respect to the Lie derivative Lie_E of the Euler vector field E: that is,
$$\mathrm{Lie}_E(\circ) = \circ, \quad \mathrm{Lie}_E(\eta) = (2-d)\eta.$$

Let $M = (M, \eta, \circ, e, E)$ be a Frobenius manifold of dimension $d \in \mathbb{C}$.

Proposition 5.

There exists a local coordinate system $(t_1, \cdots, t_\mu)$ *and a holomorphic function* $\mathcal{F} \in \mathcal{O}_M$ *such that*

- $e = \partial_1, \ \mathrm{Ker}\nabla\!\!\!/ \cong \bigoplus_{i=1}^{\mu} \mathbb{C}_M \cdot \partial_i$
- η *naturally gives a* $\mathbb{C}_M$*-bilinear* $\eta : \mathrm{Ker}\nabla\!\!\!/ \times \mathrm{Ker}\nabla\!\!\!/ \longrightarrow \mathbb{C}_M$,
- $E = \sum_{i=1}^{\mu} [(1-q_i)t_i + c_i]\,\partial_i$, *if* $q_i \neq 1$ *then* $c_i = 0$,
- $\eta(\partial_i \circ \partial_j, \partial_k) = \partial_i \partial_j \partial_k \mathcal{F}$,
- $E\mathcal{F} = (3-d)\mathcal{F} + ($*quadratic terms in* $t_2, \cdots, t_\mu)$,

where $\partial_i = \dfrac{\partial}{\partial t_i}$.

The coordinate system $(t_1, \cdots, t_\mu)$ is called a flat coordinate system, and the function $\mathcal{F}$ is called the Frobenius potential. A Frobenius structure is locally determined by a flat coordinate system and Frobenius potential:

$$\underset{(\eta, \circ, e, E)}{\text{Frobenius str.}} \overset{\text{local}}{=} \underset{(t_1, \ldots, t_\mu)}{\text{flat coordinate}} + \underset{\mathcal{F}}{\text{Frobenius potential}}$$

Define a subset $D \subset M$, called the discriminant, by

$$D := \{p \in M \mid \det(C_E)(p) = 0\},$$

where $C_E \colon \mathcal{T}_M \longrightarrow \mathcal{T}_M,\ C_E(\delta) := E \circ \delta$.

Set $M^{\mathrm{reg}} := M \setminus D$.

Definition 6 (Intersection form).

Define a symmetric $\mathcal{O}_{M^{\mathrm{reg}}}$-bilinear form $g : \mathcal{T}_{M^{\mathrm{reg}}} \times \mathcal{T}_{M^{\mathrm{reg}}} \longrightarrow \mathcal{O}_{M^{\mathrm{reg}}}$ by

$$g(\delta, \delta') := \eta(C_E^{-1}\delta, \delta').$$

We call the induced symmetric $\mathcal{O}_{M^{\mathrm{reg}}}$-bilinear form $g : \Omega^1_{M^{\mathrm{reg}}} \times \Omega^1_{M^{\mathrm{reg}}} \longrightarrow \mathcal{O}_{M^{\mathrm{reg}}}$ the intersection form of the Frobenius manifold.

On a flat coordinate system $(t_1, \cdots, t_\mu)$, the intersection form is given by

$$g(dt_i, dt_j) = \sum_{a,b=1}^{\mu} \eta^{ia}\eta^{jb} E\partial_a\partial_b\mathcal{F}, \quad \eta^{ia} := \eta(dt_i, dt_a).$$

The Levi–Civita connection ∇ with respect to g is called the second structure connection of the Frobenius manifold $(M, \eta, \circ, e, E)$. It is known that the connection ∇ is flat.

Define a local system $\mathrm{Sol}(\nabla)$ on M^{reg} by

$$\mathrm{Sol}(\nabla) := \{x \in \mathcal{O}_{M^{\mathrm{reg}}} \mid \nabla dx = 0\}.$$

Fix a point $p_0 \in M^{\mathrm{reg}}$. Then one can obtain a group homomorphism

$$\rho \colon \pi_1(M^{\mathrm{reg}}, p_0) \longrightarrow \mathrm{Aut}(\mathrm{Sol}(\nabla)_{p_0}).$$

The image $W_M := \mathrm{Im}\,\rho$ is called the monodromy group of the Frobenius manifold. Denote by $\widetilde{M}^{\mathrm{reg}}$ the monodromy covering space of M^{reg}.

By the analytic continuation, we have a holomorphic map

$$\widetilde{M}^{\mathrm{reg}} \times \mathrm{Sol}(\nabla)_{p_0} \longrightarrow \mathbb{C}, \quad (\widetilde{p}, x) \mapsto x(\widetilde{p}).$$

Put $\mathbb{E} := \mathrm{Hom}_{\mathbb{C}}(\mathrm{Sol}(\nabla)_{p_0}, \mathbb{C})$.

Definition 7 (Period mapping).

The period mapping associated to the Frobenius manifold is a holomorphic map

$$\widetilde{M}^{\mathrm{reg}} \longrightarrow \mathbb{E}, \quad \widetilde{p} \mapsto (x \mapsto x(\widetilde{p})).$$

Primitive form and period mapping

There exists 3 different constructions of Frobenius manifolds:

(A) Gromov–Witten theory,

(B) Deformation theory + Primitive form,

(Rep) Invariant theory of Weyl group.

For a generalized root system $(R, \mathbf{c})$ of type ADE, we can construct a Frobenius manifold $M_{(R,\mathbf{c})} = (\mathfrak{h}/\!\!/W, \eta, \circ, e, E)$ via (Rep).

We will see it in the last section.

Frobenius manifold via (B) for type ADE

Let $f\colon \mathbb{C}^3 \longrightarrow \mathbb{C}$ be a polynomial of type ADE.
The "universal deformation space" M_f of the holomorphic function f is isomorphic to $\mathbb{C}^\mu$.

A Frobenius structure $(\eta_\zeta, \circ, e, E)$ on M_f depends on a primitive form ζ.
For the Frobenius manifold $M_{(f,\zeta)} = (M_f, \eta_\zeta, \circ, e, E)$, the period mapping is given by the one associated to the primitive form ζ.

Moreover, an isomorphism $M_{(f,\zeta)} \cong M_{(R,\mathbf{c})}$ of Frobenius manifolds is induced by the period mapping associated to the primitive form ζ.

The construction (Rep) is known for

- Finite Weyl group
  [K. Saito, K. Saito–Yano–Sekiguchi, Dubrovin]
- (Extended) Affine Weyl group
  [Dubrovin–Zhang]
- Elliptic Weyl group
  [K. Saito, Satake, Dubrovin, Bertola]
- Complex reflection groups
  [Kato–Mano–Sekiguchi, Konishi–Minabe–Shiraishi]

Problem (K. Saito)

Develop a theory of generalized root system and a theory of flat invariants for the root system in a self-contained way, to answer the Jacobi's inversion problem for further cases of period mappings.

Remark: theory of flat invariants $\fallingdotseq$ theory of Frobenius manifold.

Bridgeland stability condition

Let $\mathcal{D}$ be a $\mathbb{C}$-linear triangulated category of finite type.

A stability condition $(Z, \mathcal{P})$ on $\mathcal{D}$ consists of

- $Z : K_0(\mathcal{D}) \longrightarrow \mathbb{C}$; group homomorphism,
- $\mathcal{P}(\phi)$: additive full sub categories ($\phi \in \mathbb{R}$),

satisfying some axioms.

Denote by $\mathrm{Stab}(\mathcal{D})$ the space of all stability conditions on $\mathcal{D}$. It is known that $\mathrm{Stab}(\mathcal{D})$ has a natural topology.

Theorem 8 (Bridgeland).

The natural forgetful map

$$\mathcal{Z} : \mathrm{Stab}(\mathcal{D}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(K_0(\mathcal{D}), \mathbb{C}), \quad (Z, \mathcal{P}) \mapsto Z,$$

is a local homeomorphism. In particular, $\mathrm{Stab}(\mathcal{D})$ *has a structure of complex manifolds.*

Conjecture 9 (Takahashi).

Let $f : \mathbb{C}^3 \longrightarrow \mathbb{C}$ *be a polynomial of type ADE and* Q *a Dynkin quiver of type corresponding to* f*. There should exist an isomorphism of complex manifolds*

$$M_f \cong \mathrm{Stab}(\mathcal{D}^b(Q)).$$

In particular, $\mathrm{Stab}(\mathcal{D}^b(Q))$ *has a Frobenius structure (and real structure).*

[Bridgeland–Qiu–Surtherland] : A_2 case.
[Haiden–Katzarkov–Kontsevich] : A_n and affine $A_{p,q}$ cases.

Moreover, they showed that the natural map

$$\mathcal{Z} : \mathrm{Stab}(\mathcal{D}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(K_0(\mathcal{D}), \mathbb{C})$$

is corresponding to the (exponential type) period mapping of a certain primitive form under the isomorphism.

1. Generalized root system and triangulated category
   - Generalized root system
   - Triangulated category
   - Geometric background
2. Frobenius manifold
   - Definition of Frobenius manifolds
   - Primitive form and period mapping
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3. From generalized root system to Frobenius manifold
   - ADE type
   - ℓ-Kronecker quiver

ADE type

Let Q be a Dynkin quiver.
In this case the Coxeter element $\mathbf{c}$ has finite order. The number $h := \mathrm{ord}(\mathbf{c}) \in \mathbb{Z}_{\geq 2}$ is the Coxeter number.

Theorem 10 (Chevalley's Theorem).

1. *The W-invariant ring $\mathbb{C}[\mathfrak{h}]^W$ of the polynomial ring $\mathbb{C}[\mathfrak{h}]$ is generated by μ homogeneous polynomials $p_1, \cdots, p_\mu$ such that*

$$h = \deg p_1 > \deg p_2 \geq \cdots \geq \deg p_{\mu-1} > \deg p_\mu = 2.$$

2. *The set of degrees $\{\deg p_1, \ldots, \deg p_\mu\}$ does not depend on the choice of $p_1, \cdots, p_\mu$.*
3. *The eigenvalues of the Coxeter element $\mathbf{c}$ are*

$$\exp\left(2\pi\sqrt{-1}\,\frac{\deg p_1 - 1}{h}\right), \cdots, \exp\left(2\pi\sqrt{-1}\,\frac{\deg p_\mu - 1}{h}\right).$$

By Chevalley's Theorem, the orbit space $\mathfrak{h}/\!\!/W$ is isomorphic to $\mathbb{C}^\mu$ as complex manifolds.

Theorem 11 (Saito, Saito–Yano–Sekiguchi, Dubrovin).

There exists a unique Frobenius structure $(\eta, \circ, e, E)$ of dimension $d = 1 - \frac{2}{h}$ on $\mathfrak{h}/\!\!/W \cong \mathbb{C}^\mu$ satisfying

1. *The intersection form g is determined by the Cartan form I.*
2. *There exist W-invariant homogeneous polynomials $t_1, \cdots, t_\mu$ such that $(t_1, \ldots, t_\mu)$ is a (global) flat coordinate system of the Frobenius manifold.*
3. *The Euler vector field E is given by*

$$E = \sum_{i=1}^{\mu} \frac{\deg t_i}{h} t_i \frac{\partial}{\partial t_i}.$$

Define the regular subset $\mathfrak{h}^{\mathrm{reg}}$ of $\mathfrak{h}$ by

$$\mathfrak{h}^{\mathrm{reg}} := \mathfrak{h} \setminus \bigcup_{\alpha \in \Delta^{\mathrm{re}}} H_\alpha,$$

where $H_\alpha := \{x \in \mathfrak{h} \mid \langle \alpha, x \rangle = 0\}$ is the root hyperplane of $\alpha \in \Delta^{\mathrm{re}}$.

For the Frobenius manifold $(\mathfrak{h}/\!\!/W, \eta, \circ, e, E)$, we can consider the regular locus of $\mathfrak{h}/\!\!/W$. This is naturally isomorphic to the W-orbit space of $\mathfrak{h}^{\mathrm{reg}}$.

$$(\mathfrak{h}/\!\!/W)^{\mathrm{reg}} \cong \mathfrak{h}^{\mathrm{reg}}/W$$

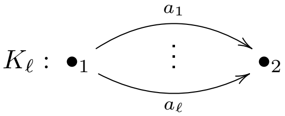
For an A_n-quiver, Ikeda showed that the universal covering space $\widetilde{(\mathfrak{h}/W)}^{\mathrm{reg}}$ with the period mapping can be identified with the space of stability conditions of "the derived category of N-Calabi–Yau completion of A_n-quiver $\check{\mathcal{D}}_N(A_n)$" with the central charge map:

$$\begin{array}{ccc} \mathrm{Stab}^\circ(\check{\mathcal{D}}_N(A_n)) & \xrightarrow{\cong} & \widetilde{(\mathfrak{h}/W)}^{\mathrm{reg}} = \widetilde{\mathfrak{h}^{\mathrm{reg}}} \\ \downarrow \mathcal{Z} & \circlearrowleft & \downarrow \text{period} \\ \mathfrak{h} = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{C}) & \xrightarrow{\cong} & \mathbb{E} \end{array}$$

where $^\circ$ denotes a connected component.

ℓ-Kronecker quiver

Let $Q = K_\ell$ be the ℓ–Kronecker quiver:



The simple representations S_1 and S_2 induces a basis on $K_0(\mathcal{D}^b(K_\ell))$. Then the matrix representation C_ℓ of the Cartan form I with respect to the basis $\{[S_1], [S_2]\}$ is given by

$$C_\ell = \begin{pmatrix} 2 & -\ell \\ -\ell & 2 \end{pmatrix}.$$

This is nothing but the generalized Cartan matrix associated with the ℓ–Kronecker quiver K_ℓ.

Hence, one can consider the Kac–Moody Lie algebra $\mathfrak{g}$ associated with the generalized Cartan matrix C_ℓ.

- When $\ell = 1$, the Kac–Moody algebra $\mathfrak{g}$ is of finite type (A_2),
- When $\ell = 2$, the Kac–Moody algebra $\mathfrak{g}$ is of affine type (affine A_1),
- When $\ell \geq 3$, the Kac–Moody algebra $\mathfrak{g}$ is of indefinite type.

In what follows, we assume that $\ell \geq 3$.

The Coxeter element $\mathbf{c}$ does not have finite order. We need to modify the definition of the Coxeter number h.

Let ρ be the spectral radius of the Coxeter element $\mathbf{c}$:

$$\rho = \frac{\ell^2 - 2 + \sqrt{\ell^4 - 4\ell^2}}{2} \ (> 1).$$

The eigenvalues of c are ρ and ρ^{-1}. It can be regarded as follows:

$$\rho = \exp\left(2\pi\sqrt{-1}\frac{\log\rho}{2\pi\sqrt{-1}}\right), \quad \rho^{-1} = \exp\left(-2\pi\sqrt{-1}\frac{\log\rho}{2\pi\sqrt{-1}}\right).$$

Define a number $h \in \mathbb{C}\backslash\mathbb{R}$ by $h := \dfrac{2\pi\sqrt{-1}}{\log\rho}$ and hence we have

$$\rho = \exp\left(2\pi\sqrt{-1}\frac{2-1}{h}\right), \quad \rho^{-1} = \exp\left(2\pi\sqrt{-1}\frac{h-1}{h}\right).$$

We should consider W-invariant function t of degree $\deg t = h$.

Define the set of imaginary roots Δ_+^{im} by

$$\Delta_+^{\mathrm{im}} := \{w(\alpha) \in L \mid w \in W,\ \alpha \in L_+ \text{ s.t. } I(\alpha, \alpha_i) \leq 0,\ i = 1, 2\}.$$

and the imaginary cone $\mathcal{I} \subset \mathfrak{h}_{\mathbb{R}}^*$ by the closure of the convex hull of $\Delta_+^{\mathrm{im}} \cup \{0\}$.

Definition 12.

Define an open subset $X \subset \mathfrak{h}$ by

$$X := \mathfrak{h} \setminus \bigcup_{\lambda \in \mathcal{I} \setminus \{0\}} H_\lambda$$

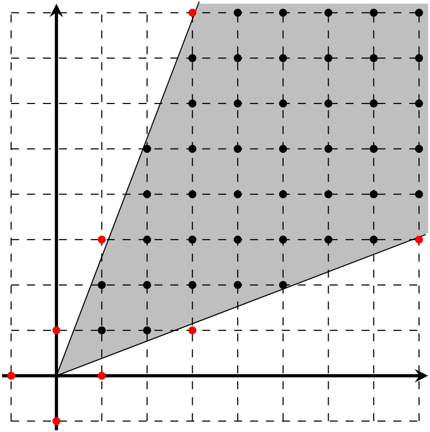
and a regular subset $X^{\mathrm{reg}} \subset X$ by

$$X^{\mathrm{reg}} := X \setminus \bigcup_{\alpha \in \Delta^{\mathrm{re}}} H_\alpha,$$

where $H_\lambda := \{Z \in \mathfrak{h} \mid Z(\lambda) = 0\}$ for $\lambda \in \mathfrak{h}^*$.

It is known that the fundamental group is $\pi_1(X) \cong \mathbb{Z}$.

Example: imaginary cone for the case $\ell = 3$:



red dots = real roots Δ^{re},
black dots = positive imaginary roots Δ_{+}^{im},
gray shaded domain = imaginary cone $\mathcal{I}$.

There exists a W-action on the universal covering space $\widetilde{X}$ such that the map $\widetilde{X} \longrightarrow X$ is W-equivariant.

Definition 13.

Define a complex analytic space $\widetilde{X}/\!\!/W$ as follows:

- *The underlying space is the quotient space $\widetilde{X}/W$ and denote by $\pi : \widetilde{X} \to \widetilde{X}/W$ the quotient map.*
- *The structure sheaf is $\mathcal{O}_{\widetilde{X}/\!\!/W} := \pi_* \mathcal{O}_{\widetilde{X}}^W$, where $\mathcal{O}_{\widetilde{X}}^W$ is the W-invariant subsheaf of $\mathcal{O}_{\widetilde{X}}$.*

Proposition 14.

The orbit space $\widetilde{X}/\!\!/W$ has a structure of complex manifolds. Moreover, there exists an isomorphism

$$\widetilde{X}/\!\!/W \cong \mathrm{Stab}(\mathcal{D}^b(K_\ell)) \cong \mathbb{C} \times \mathbb{H}$$

as complex manifolds.

The following theorem is our main result.

Theorem 15 (Ikeda-O-Shiraishi-Takahashi).

There exists a unique Frobenius structure $(\eta, \circ, e, E)$ *of dimension* $d = 1 - \frac{2}{h}$ *on* $\widetilde{X}/\!\!/W$ *satisfying*

1. *The intersection form* g *is determined by the Cartan form* I.
2. *There exist* W*-invariant homogeneous functions* t_1 *and* t_2 *such that* $(t_1, \ldots, t_\mu)$ *is a (local) flat coordinate system of the Frobenius manifold.*
3. *The Euler vector field* E *is given by*

$$E = t_1 \frac{\partial}{\partial t_1} + \frac{2}{h} t_2 \frac{\partial}{\partial t_2} = \frac{\deg t_1}{h} t_1 \frac{\partial}{\partial t_1} + \frac{\deg t_2}{h} t_2 \frac{\partial}{\partial t_2}.$$

Thank you for your attention !